Type Systems

CS 6340

{HEADSHOT}

This lesson introduces a popular kind of static analysis called type systems.

Type systems are often specified as part of the programming language, and built into compilers or interpreters for the language.

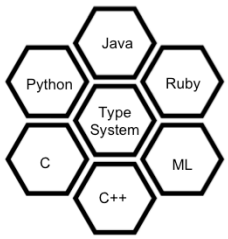
The main purpose of a type system is to reduce the possibility of bugs by checking for logic errors. A common example of such an error is applying an operation to operands that does not make sense, such as adding an integer to a string in a Java program.

A type system checks for such errors using a collection of rules which assign types to different parts of the program, such as variables, expressions, and functions.

We will learn about the notation for type systems, various useful properties of type systems, and how type systems can even be used for describing other kinds of static analyses.

Type Systems

- Most widely used form of static analysis
- Part of nearly all mainstream languages
  - Important for quality



Type systems are the most widely used form of static analysis.

They are used in nearly all mainstream programming languages; one reason for this is that they are widely recognized as being important for maintaining the quality of programs.

Motivation

```
1: class T {
2:   int f(float a, int b,
3:         int[] c) {
4:     if (a)
5:       return b;
6:     else
7:       return c;
8:   }
9: }
```

File `T.java`

```
prompt$ javac T.java

T.java:4: error: incompatible types
         if (a)
             ^
    required: boolean
    found:    float
T.java:7: error: incompatible types
             return c;
                    ^
    required: int
    found:    int[]
2 errors
```

Let's look at an example of the kind of errors that a type system typically catches.

Consider the following Java program which defines a function f in class T in a file named T.java. When we compile this program using the java compiler, it produces two type errors:

- The first error states that the if-condition on line 4 expects its argument a to be of type boolean, but it is instead of type float.
- The second error states that the return result c of the function f on line 7 is expected to be of type int, since the specified return type of function f is int, but it instead has type integer array.

Java's type system prevents logic errors like these which could cause the program to produce undesired results.

Type Systems

- Most widely used form of static analysis
- Part of nearly all mainstream languages
  - Important for quality
- Provides notation useful for describing static analyses:
  type checking, dataflow analysis, symbolic execution, ...

Besides being a static analysis in their own right, type systems also provide notation which is useful for describing a variety of different static analyses, such as type checking, dataflow analysis, symbolic execution, and more.

What Is a Type?

- A type is a set of values
- Examples in Java:
  - `int` is the set of all integers between `-2^31` and `(2^31)-1`
  - `double` is the set of all double-precision floating point numbers
  - `boolean` is the set {`true`, `false`}

To discuss type systems, we must first ask ourselves, what do we mean by a "type"? It is easy to label the number 2 an integer and the number 1.125 a floating-point number, but it isn't necessarily obvious how to define what a "type" is.

To put it simply, a "type" is a set of values that constitute that type. For example, in Java, the type "int" is the set of all integers (at least, those integers that can be represented in 32 bits of data); the "double" type is the set of real numbers that can be represented in a certain 64-bit IEEE standard, and the "boolean" type is the set consisting of the values "true" and "false."

More Examples

- `Foo` is the set of all objects of class `Foo`
- `List<Integer>` is the set of all Lists of Integer objects
  - `List` is a type constructor
  - `List` acts as a function from types to types
- `int -> int` is the set of functions taking an int as input and returning another int
  E.g.: increment, a function that squares a number, etc.

Moving away from primitive types in Java, we can come up with more complex examples.

If we have a class named Foo, then the Foo type is the set of all objects of class Foo.

List<Integer> is the set of all List objects which contain Integer objects. Note that List is a type constructor: it can be treated as a function that takes one type (in this case, the Integer type) and maps it to another type (in this case, the type of Lists of Integers).

We can also have function types. For example, int arrow int is the set of all functions that take an int as their argument and return an int as their result. Some examples of functions of this type would be the increment operation, a function that squares its input, and many others.

Abstraction

- All static analyses use abstraction
  - Represent sets of concrete values as abstract values
- Why?
  - Can't directly reason about infinite sets of concrete values (wouldn't guarantee termination)
  - Improves performance even in case of (large) finite sets
- In type systems, the abstractions are called types

Abstractions are an integral part of all forms of static analysis. Recall that in dataflow analysis, we abstract the control flow of programs, and in pointer analysis, we additionally abstract the heap by lumping together objects allocated at the same allocation statement in the program.

Why do we do this? The reason for abstracting program properties goes back to the undecidability of static analysis: we cannot reason directly about the infinite set of all possible concrete values (if we tried to, we have no guarantee that our analysis would terminate). Additionally, even for finite sets of concrete values, abstraction allows us to improve the performance of our analysis.

The abstractions used in type systems are the types themselves: each type represents a large, possibly infinite set of concrete values. By abstracting away these large sets, we enable ourselves to reason about the values based on properties they have in common.

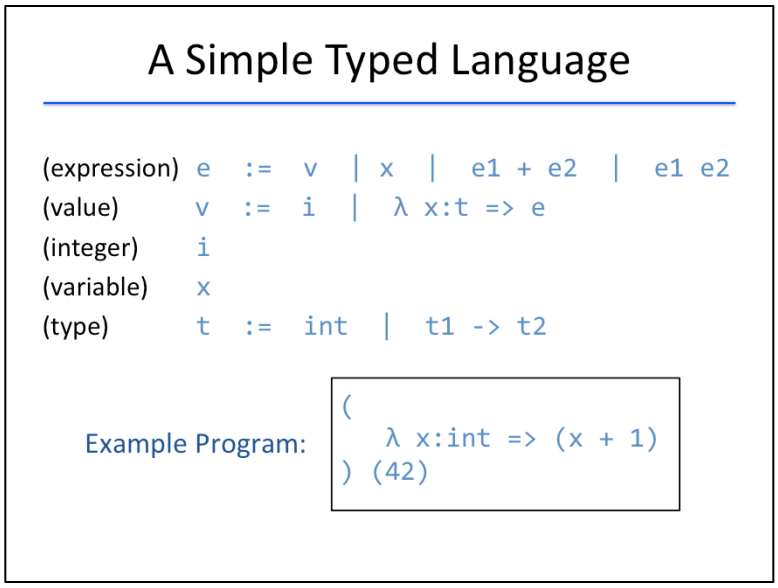
What Is a Type?

- A type is an example of an abstract value
  - Represents a set of concrete values
- In type systems:
  - Every concrete value is an element of some abstract value

  => every concrete value has a type

A type is an example of what is known as an abstract value in that it represents a set of concrete values: the Java type int represents all integers representable with 32 bits, and the List type represents the set of all List objects.

In type systems, every concrete value must be an element of some abstract value; therefore, every concrete value has a type.



Let's introduce a simple typed language based on the lambda calculus. We will use this language to provide concrete examples for the abstract principles of type systems.

Each expression in this language ultimately evaluates to one of two kinds of values v: integers (denoted here by i, but which may be any integer value) and functions denoted as follows [point to "lambda x: t => e"], lambda x of type t yields e (where "x of type t" is represented by x followed by a colon followed by t, and "yields" is represented by an = sign followed by a > sign). This notation means a function which takes an argument x of type t, and returns the result of computing expression e.

Both kinds of values -- integers and functions -- can be stored in variables denoted by x. (Other identifiers may be used for variables as well.)

The plus operation requires that both e1 and e2 be integer-valued expressions. Our type system will ensure that this requirement is indeed met in all executions of a program in this language; in other words, that e1 and e2 can never evaluate to a function value.

Similarly, this operation [point to e1 e2] denotes a call to a function e1 with argument e2, and there is a similar requirement on the kinds of values that e1 and e2 might evaluate to. Our type system will also ensure this requirement.

To meet these requirements, the system consists of two types, corresponding to the two kinds of values in our language: an int type, which is the type of all integer values, and a function type denoted by t1 arrow t2 [point to t1 -> t2], where the arrow is represented by a hyphen followed by a greater-than sign) which is the type of functions.

Note that the int type is a base type, whereas function type is a compound type, as we will illustrate using an example program in this language.

This example defines a function that takes an integer x as an argument and returns x+1. The type of this expression is int -> int [handwrite this]. Then this function is applied to the integer 42, yielding the integer 43.

QUIZ: Programs and Types

Write the type of each program, or NONE if it is not typeable:

Program	Type
λ x:int => (x + x)	
(λ x:int => (x + x)) (10)	
42 (λ x:int => (x + 5))	
λ x:int => (λ y:int => (x + y))	
(λ x:int => x) + 10	

{QUIZ SLIDE}

To check your understanding of this language, let's do a quiz. Here we've listed five short programs:

- lambda x of type int yielding x plus x
- lambda x of type int yielding x plus x, applied to 10
- 42 applied to lambda x of type int yielding x plus 5
- lambda x of type int yielding (lambda y of type int which in turn yields x plus y)
- lambda x of type int yielding x, plus 10

Write the type of each program in the box beside it, or NONE if it is not typeable. Note that, in our type system, each typeable program is guaranteed to have a unique type, so there will not be any ambiguity in the type of these expressions; please use parentheses to disambiguate the type if needed.

QUIZ: Programs and Types

Write the type of each program, or NONE if it is not typeable:

Program	Type
λ x:int => (x + x)	int -> int
(λ x:int => (x + x)) (10)	int
42 (λ x:int => (x + 5))	NONE
λ x:int => (λ y:int => (x + y))	int -> (int -> int)
(λ x:int => x) + 10	NONE

{SOLUTION SLIDE}

The first program, lambda x of type int yields x plus x, is a function that takes an integer and yields the expression x plus x (it is a function that doubles its input). The type of this expression is therefore int arrow int.

The second program takes the function in the first program and applies it to the number 10. The result of this program is 20, which has the type int.

The third program is not typeable as it violates a requirement of function calls, namely, that the first expression in the pair must be a function, whereas here it is an integer 42.

The next program is a bit trickier: it takes an integer x and returns a function that takes a number and adds it to x. So the type for the whole expression is the compound type int arrow (int arrow int), where we put parentheses around the latter two ints to disambiguate the type. (Putting parentheses around the first two ints would be the type of a function which takes a function from int to int, and returns an int.)

The last program is not typeable as it violates a requirement of the plus operation, namely, that both its operands be integers, whereas here one of the operands is a function.

The Next Steps

- Notation for Type Systems
- Properties of Type Systems
- Describing Other Analyses Using Types Notation

Now that we have a better understanding of what types are, we will learn how to compute with them: that is, how to actually analyze programs using type systems.

Type systems have a well-developed notation for expressing these sorts of analyses. In fact, a major part of defining a type system involves defining a set of rules showing how to check that a given program has a given type.

In the next several slides, we will provide the notation for type systems, which consists of a set of rules that can be used to check that a given program has a given type. Later on in the lesson, we will discuss important properties of type systems, including how to infer the type automatically for a given program. Finally, we will see how to use the notation of types to describe other static analyses.

Notation for Inference Rules

- Inference rules have the following form:

  If (hypothesis) is true, then (conclusion) is true

- Type checking computes via reasoning:

  If `e1` *is an* `int` *and* `e2` *is a* `double`, *then* `e1*e2` *is a* `double`

- We will develop a standard notation for rules of inference

If you've studied formal logic, you may be familiar with logical implications: "if-then" statements. Inference rules in type systems are "if-then" statements which are used to reason about the types of program fragments, such as variables, expressions, and functions.

A rule of inference takes the form, "If (hypothesis) is true, then (conclusion) is true," where the hypothesis and the conclusion are logical formulas.

An example of an inference rule that might be used in a type system is: "if e1 is an int and e2 is a double, then e1 * e2 is a double."

Now let's start developing a standard notation for expressing inference rules.

From English to Inference Rule

- Start with a simplified system and gradually add features

- Building blocks:
  - Symbol $\wedge$ is "and"
  - Symbol $\Rightarrow$ is "if-then"
  - `x : t` is "`x` has type `t`"

To build this notation, we'll begin with a few small building blocks and then gradually add features as needed.

The building blocks we will use are:

- the inverted-V to represent the logical "and" operation, also called the "conjunction" operator
- the rightward-pointing arrow to represent logical implication (that is, the "if-then" operation), and
- the expression x-colon-t to represent the statement "x has type t"

When reading logical formulas aloud, the inverted-V can be read as the word "and", and the arrow symbol can be read as the word "implies."

From English to Inference Rule

- If `e1` has type `int` and `e2` has type `int`, then `e1 + e2` has type `int`
- (`e1` has type `int` ∧ `e2` has type `int`) ⇒ `e1 + e2` has type `int`
- `(e1 : int ∧ e2 : int) ⇒ e1 + e2 : int`

Let's see how we can express the following English sentence in our notation for rules of inference.

"If e1 has type int and e2 has type int, then e1+e2 has type int."

First, we can group the first two type declarations together with an inverted-V to represent their conjunction by the logical "and".

Then we can replace the "if X then Y" construction with "X arrow Y," where X is the conjoined hypotheses and Y is the conclusion "e1+e2 has type int".

Finally, we can replace the phrases "X has type int" with the notation "X colon int", giving us the following inference rule:

"e1 has type int and e2 has type int implies e1 plus e2 has type int"

From English to Inference Rule

The statement

$$(e1 : int \wedge e2 : int) \Rightarrow e1 + e2 : int$$

is a special case of

$$Hypothesis_1 \wedge \ldots \wedge Hypothesis_N \Rightarrow Conclusion$$

Observe that in this formulation of the statement, we have two separate hypotheses which have been conjoined to form a compound hypothesis. Each separate hypothesis must be true in order to guarantee that the conclusion is true.

In general, inference rules have the form

Hypothesis 1 and Hypothesis 2 and so-forth upto Hypothesis N implies Conclusion.

An inference rule may have an arbitrary (but finite) number of hypotheses that must be fulfilled to guarantee the conclusion.

Notation for Inference Rules

- By tradition, inference rules are written

$$\frac{\vdash \text{Hypothesis}_1 \quad \ldots \quad \vdash \text{Hypothesis}_N}{\vdash \text{Conclusion}}$$

- Hypotheses and conclusion are type judgments:

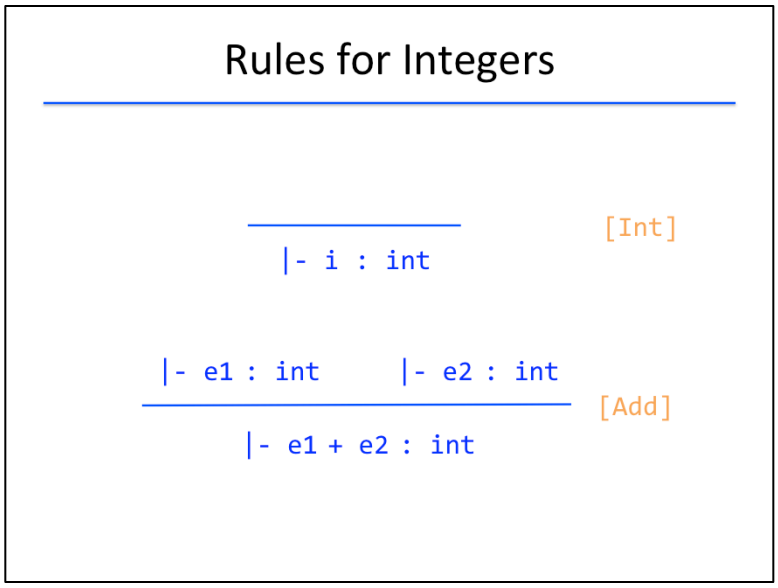
```
|- e : t
```

- `|-` means "it is provable that…"

Traditionally, inference rules are written in the following way.

Each inference rule has one conclusion and zero, one, or more hypotheses. The hypotheses are considered to be conjoined ("and"-ed) together: they must all be true in order to derive the conclusion. Inference rules without any hypotheses are called "axioms": their conclusions are statements we take to be fact without any preconditions.

The hypotheses and conclusions all take the following form [point to "|- e : t"] where e is an expression and t is a type. This statement is called a "type judgment." The pipe symbol followed by a hyphen is a symbol denoting the phrase, "it is provable that."



Let's start building the type system for our example language by focusing on the rules for integers.

The first rule we establish is that it is provable that i has type int for all literal integers i. For example we could replace the letter i with 0, 1, 200, -3, etc. and the rule would still hold. Strictly speaking, we refer to such a template rule as a schema. And because there are no hypotheses in this rule, it is an axiom schema. We will name this schema [Int] for future reference.

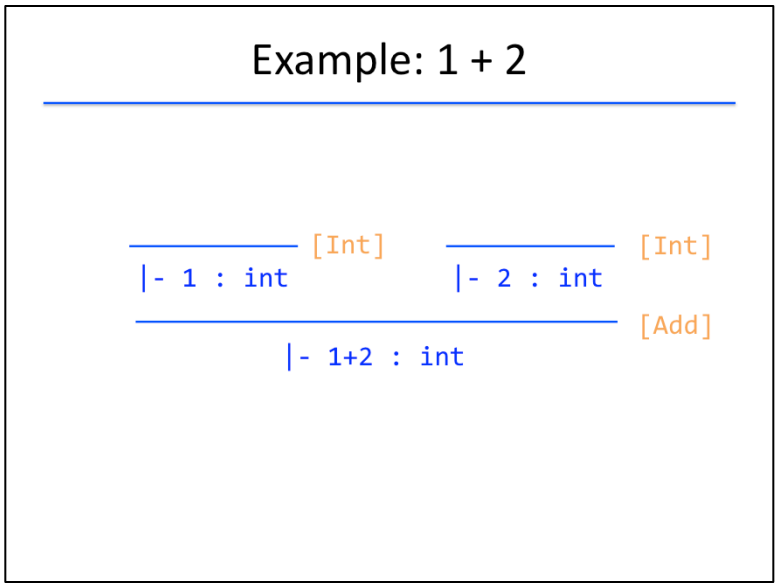
The second rule we will establish is that if it is provable that e1 has type int and that e2 has type int, then it is provable that e1 + e2 has type int. This is an inference rule schema (as opposed to an axiom schema) because there are more than 0 hypotheses. We will name this schema [Add].

Rules for Integers

- Templates for how to type integers and sums
- Filling in templates produces complete typings
- Note that:
  - Hypotheses state facts about sub-expressions
  - Conclusions state facts about entire expression

These rules thus serve as templates describing how to type integers and expressions containing the addition operator. By filling in or instantiating the templates, we can produce complete typings for expressions.

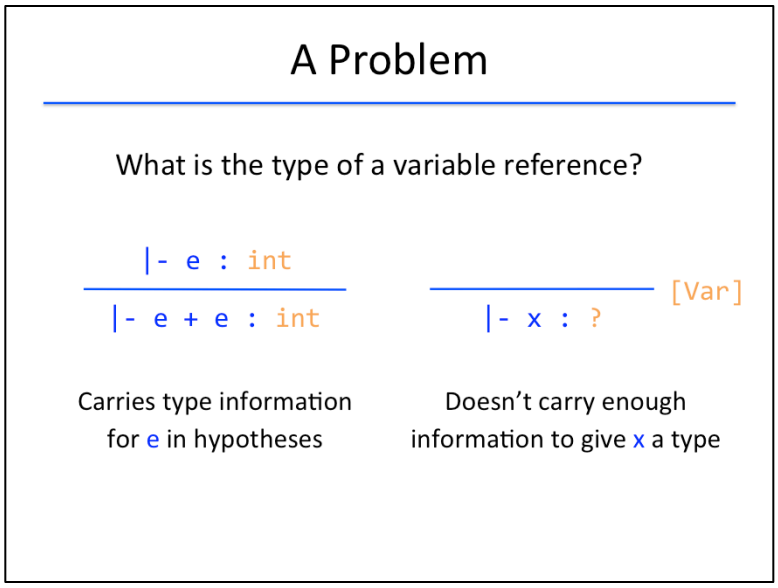
Note that while hypotheses state facts about subexpressions, conclusions state facts about the entire expression.



Here's an example of a type derivation for typing the expression 1 + 2.

We first instantiate the axiom schema [Int], filling in the number 1 for i, to produce the axiom "it is provable that 1 has type int." Similarly, we instantiate the [Int] axiom schema again to produce the axiom "it is provable that 2 has type int."

Then, we can take these two conclusions and "plug them in" as hypotheses for the inference rule schema [Add]. Because it is provable that 1 has type int and 2 has type int, it is provable that 1+2 has type int. This conclusion completes the derivation of the typing for the expression 1+2.

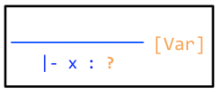


We've discussed typing for integer expressions (which, in our simple language, are restricted to sums). Using the rules we've established so far, we can derive a rule such as this one: if it is provable that e has type int, then it is provable that e+e has type int. This schema carries the type information for e in its hypotheses.

Now, we also want to apply typing rules for variables. However, if we try to axiomatically define x's type, we find that the axiom has no way of carrying the type information for x (and we don't want to impose a rule that all variables must be of a single type). Moreover, we can't provide type information via hypotheses, because these hypotheses must eventually be derived from axiomatic statements. So we've run into a problem. How can we provide appropriate type information for a variable?

A Solution

- Put more information in the rules!



- An environment gives types for free variables
  - A variable is free in an expression if not defined within the expression; otherwise it is bound
  - An environment is a function from variables to types
  - May map variables to other abstract values in different static analyses

The solution for our type system is to add more information to the rules.

We can use an environment to give type information to free variables, such as x in the rule [Var].

A variable is free in an expression if it is not defined within the expression itself. In contrast, a variable that is defined in an expression is called a bound variable. For instance, consider the expression "lambda x:int => (x + y)" [handwrite this]. Variable y is free in this expression but variable x is bound.

In a type system, an environment is a function which maps variables to types. (For example, if we declare the variable x to be an int in a Java program, the environment would map the variable x to the type int.)

The concept of an environment is general: in other kinds of static analyses, environments may map variables to other abstract values instead of types.

Type Environments

- Let A be a function from variables to types
- The sentence `A |- e : t` means:

  "Under the assumption that variables have types given by A, it is provable that expression `e` has type `t`."

Let's use the letter A to represent our environment; that is, our function from variables to types.

We'll use the notation "A pipe hyphen e colon t" to mean: "under the assumption that variables have the types given by A, it is provable that expression e has type t".

Modified Rules

- The type environment is added to all rules:

$$\frac{}{A \vdash i : int}\ [Int]$$

$$\frac{A \vdash e1 : int \qquad A \vdash e2 : int}{A \vdash e1 + e2 : int}\ [Add]$$

Let's modify our previous rules [Int] and [Add] to include the type environment before the pipe-hyphen symbol.

A New Rule

- And we can write new rules:

$$\frac{}{A \vdash x : A(x)}\ [\text{Var}]$$

And now we can write new rules, such as this axiom schema for typing variables, which we call schema [Var].

“Under environment A, it is provable that x has type A(x)”. (A(x) refers to the type assigned to x under the environment mapping A.)

Rules for Functions

$$\frac{A_{[x \mapsto t]} \vdash e : t'}{A \vdash \lambda\, x{:}t \Rightarrow e : t \to t'} \text{[Def]}$$

$A[x \mapsto t]$ means "A modified to map x to type t"

$$\frac{A \vdash e1 : t1 \to t2 \qquad A \vdash e2 : t1}{A \vdash e1\ \ e2 : t2} \text{[Call]}$$

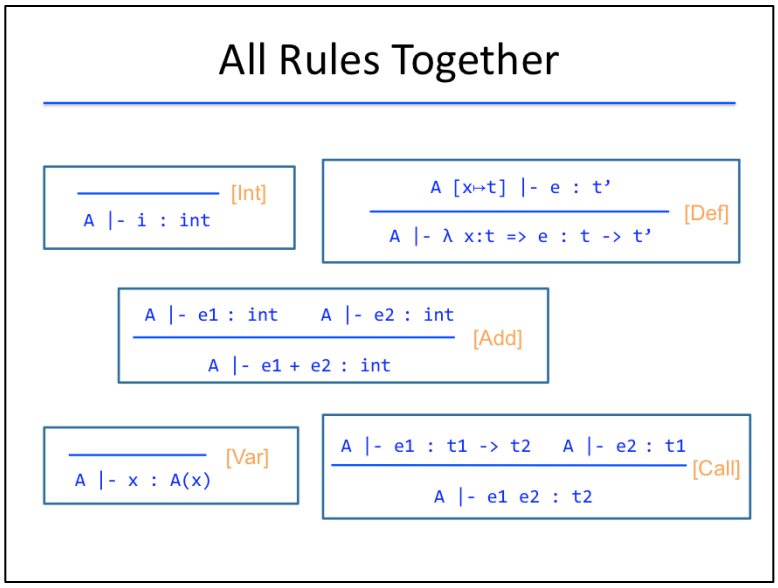
Now we can make the last of the inference rule schemas for our language. We need to introduce rules for typing function definitions and function calls. For function definitions, we make the following rule, which we call [Def]:

Using our environment A, if it is provable that e is of type t-prime, then it is provable that the function, lambda x of type t yields e, is a function of type (t arrow t-prime).

Except, we have a problem. In this expression, x is a free variable, and we need its type. We can fix this by explicitly requiring the environment to map x to type t. We do so by adding the notation x arrow t in brackets beside the letter A in the hypothesis of this rule. Notice that this environment is the same as that in the conclusion of the rule, except that the environment in the conclusion of the rule does not give the type of x. This is correct since x is not a free variable in this expression: it is bound, or defined as the argument of this function.

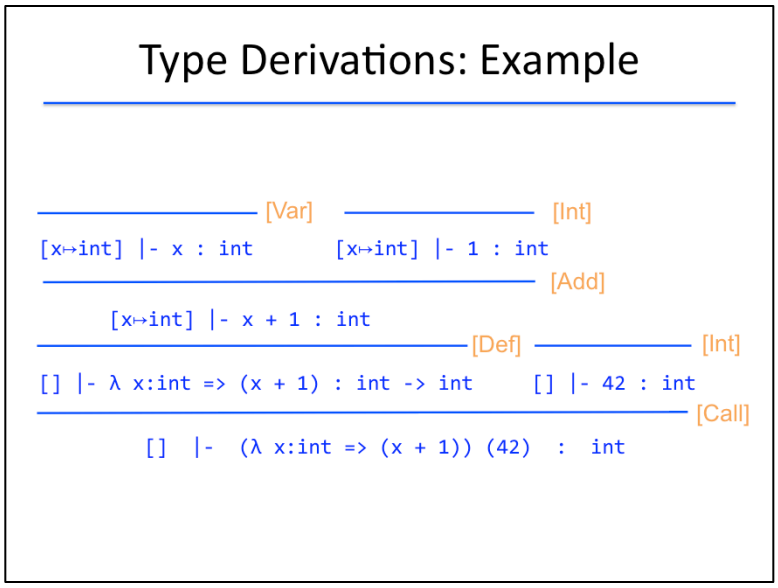
Finally, we define the inference rule schema [Call] to give typing information for applications of functions to other expressions.

If, under the environment A, it is provable that e1 has type (t1 arrow t2) and e2 has type t1, then it is provable that the result of calling e1 with argument e2 has type t2.



This concludes the set of schemas we need in order to completely type our example lambda-calculus language. Notice that there is exactly one schema for each of the five syntactic expression forms in this language.

Now let's look at how we would derive the typing for a nontrivial expression using these 5 schemas. Later, we will see how to algorithmically compute these derivations, but let's work through some by hand first to get a feel for them.



A type derivation is a procedure which allows us to derive the type of a program, if one exists. The goal of the derivation is to show that an expression has a given type derived, ultimately, from axiomatic statements (that is, there are no infinite or cyclic chains of hypotheses needed to show that some statement is true).

In this slide we will use our five schemas to derive the type of the program “lambda x of type int yields x+1, applied to 42” shown on the bottom of this slide. (See the instructor notes for a list of all the schemas for this language.)

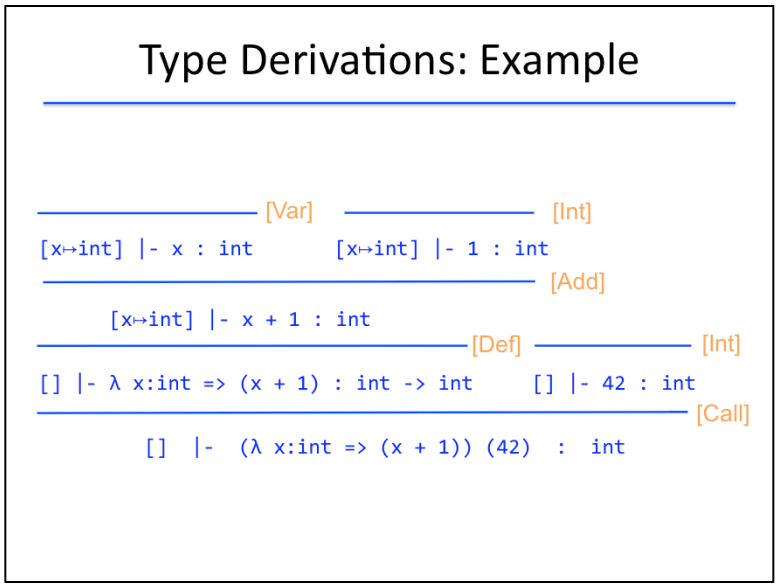
We will work from the bottom up, checking to ensure that each single derivation fits the template for one of the rules we previously defined.

The [Call] rule schema requires that if applying expression e1 to expression e2 yields type t2, then expression e2 must be of type t1, and expression e1 must be a function mapping t1 to t2. In this case, the lambda expression plays the role of expression e1, the number 42 plays the role of expression e2, and t1 and t2 are both the int type. The [Call] rule schema is therefore satisfied.

Notice that we maintain an empty environment for this template: there are no variables mapped to types. We use an empty array to represent this environment with no such mappings.

Now let’s derive the types of the hypotheses of the [Call] rule. The [Int] axiom schema allows us to assert that the literal 42 has type int without requiring any further hypotheses.

If we look at the first hypothesis of the [Call] rule, we see we have a function definition. We therefore check to see that the [Def] rule schema can be applied to this definition. This schema requires that if the conclusion is a function mapping a variable x of type t1 to an expression of type t2, then the hypothesis must be that it is provable that the output expression is of type t2 under the same environment with a single additional mapping of the variable x to type t1.



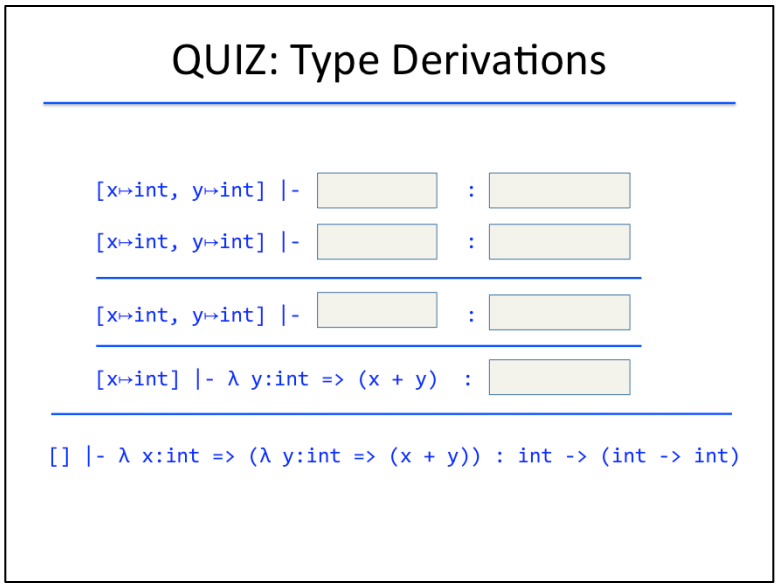
In this case, notice that the environment in the conclusion is empty and the environment in the hypothesis maps the variable x to the type int. Under this assumption, the hypothesis states that it is provable x+1 (the expression the function outputs) is an int. Therefore, the [Def] rule schema is correctly instantiated, and we now proceed to prove that x+1 has type int under the environment where x is mapped to type int.

x+1 is an addition expression, so we now check that the [Add] rule schema holds. [Add] requires that the hypotheses and conclusion have the same environment, that the expression in the conclusion have type int, and that it be provable the expressions in the hypotheses also have type int. The schema has been correctly filled in, so now we must verify that each of the hypotheses are provable.

Like before, we have the [Int] axiom schema, under which the literal 1 is provable to be an int regardless of the environment. It is not a problem that x is in the environment but doesn't appear in the expression; we may still conclude that 1 is provably of type int.

We therefore focus on the last unproven statement: it is provable that x is of type int under the environment where x is mapped to type int. However, we have a different axiom schema, [Var], under which we may prove that a variable has type t under the environment where that variable is mapped to type t. Because the [Var] axiom schema may be instantiated by filling in t by the type int, the statement that x has type int is provable.

We are now done: all subexpressions of the entire program have been shown to be provable from axioms, leaving no unproven hypotheses. Therefore, we have completed the derivation, and proven that the full program has type int.



{QUIZ SLIDE}

Now let's do a quiz to let you practice a type derivation on your own.

Recall that in the previous quiz, you answered that the program "lambda x of type int yields the function (lambda y of type int yields x plus y)" has the type of a function mapping ints to (functions mapping ints to ints).

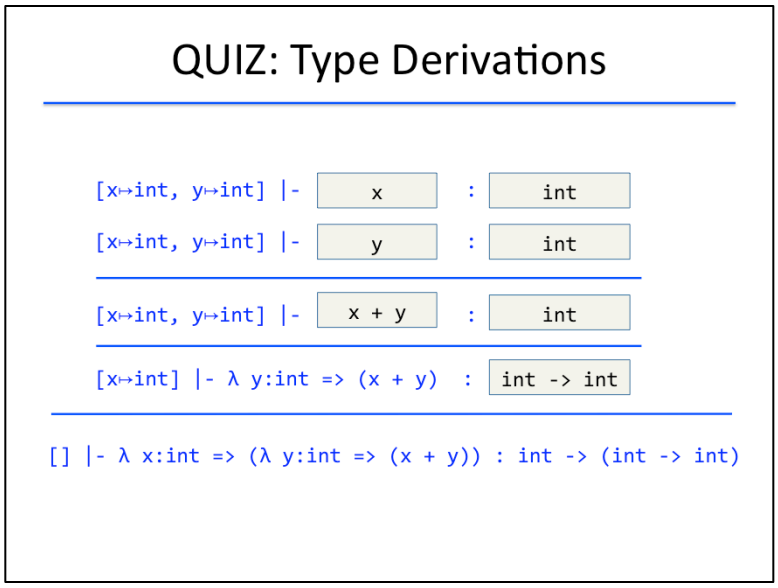
Let's suppose we wish to prove this type judgment under an empty environment. We take this to be the conclusion of a single derivation with the following hypothesis: "Under the environment mapping x to type int, it is provable that the function, lambda y of type int yields x plus y, has type BLANK."

This hypothesis is the conclusion of another derivation whose hypothesis is: "Under the environment mapping x to type int and y to type int, it is provable that BLANK has type BLANK."

This statement is the conclusion of yet another derivation with two hypotheses: "Under the environment mapping x to type int and y to type int, it is provable that BLANK has type BLANK," and "Under the environment mapping x to type int and y to type int, it is provable that BLANK has type BLANK."

The two hypotheses in this topmost rule are derived from the application of axioms which are not shown for brevity.

Complete the type derivation by filling in each of the 7 blanks with appropriate expressions and types. If you need them as a reference, the five typing schemas for this language are shown in the instructor notes.



{SOLUTION SLIDE}

Let's again work from the bottom up. We need to apply the [Def] inference rule schema to the last step in the derivation. This means we need the mapping x to int in the environment of the hypothesis, and we now need to prove that the type of the output expression is int -> int.

The expression, lambda y of type int yields x + y, is now the conclusion of another derivation, which is another instance of the [Def] inference rule schema. We add the mapping y to int to the environment in the hypothesis, and it is now required to prove that the expression x+y has type int.

Finally, to prove that x+y has type int, we need to prove that under the environment x maps to int and y maps to int, x has type int and y has type int. This is an application of the [Add] inference rule schema.

This completes the exercise, but note that, to complete this derivation, we would then need to use the [Var] axiom schema twice, once to derive that x has type int and again to derive that y has type int.

Back to the Original Example

```
1: class T {
2:   int f(float a, int b,
3:         int[] c) {
4:     if (a)
5:       return b;
6:     else
7:       return c;
8:   }
9: }
```

File `T.java`

```
prompt$ javac T.java

T.java:4: error: incompatible types
         if (a)
             ^
    required: boolean
    found:    float
T.java:7: error: incompatible types
             return c;
                    ^
    required: int
    found:    int[]
2 errors
```

We are now done with the notation for type systems. Next, we will see some properties of type systems, such as:

- What guarantees we have if a program type-checks (that is, it is successfully typed according to the axiom and inference rule schemas) versus if the program does not.
- Algorithms to mechanically compute a type derivation, if it exists.

Let's return now to our Java example from the beginning of the lesson. In this case, the function f checks whether the variable `a` evaluates to true; if so, then it returns `b`; otherwise, it returns `c`. Our lambda-calculus language from earlier currently does not have the capacity to directly express this behavior, so let us extend the language to allow if-then-else statements. We'll need to add a new inference rule schema to check that if-then-else statements are correctly typed.

A More Complex Rule

$$\frac{\begin{array}{l}A \vdash e0 : bool \\ A \vdash e1 : t1 \\ A \vdash e2 : t2 \\ \quad t1 = t2\end{array}}{A \vdash \text{if } e0 \text{ then } e1 \text{ else } e2 : t1}\ \text{[If-Then-Else]}$$

We'll use this rule to illustrate several ideas . . .

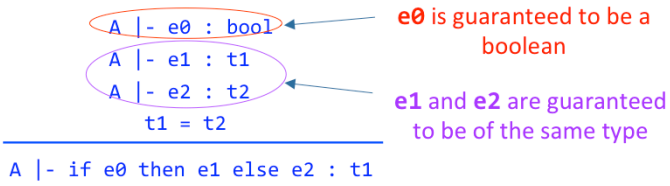
The rule we'll add, which we'll call [If-Then-Else], has four hypotheses. If under the environment A it is provable that e0 has type bool, e1 has type t1, e2 has type t2, and t1 equals t2, then it is provable that the expression "if e0 then e1 else e2" has type t1.

Note that we're implicitly introducing a new type to our language, the bool type. We won't formally add all the required schemas to type boolean expressions in our language, but you can try writing these rules yourself as an exercise. For the purposes of this lesson, we will assume boolean expressions to be fully typed going forward.

Also notice we have a new form of hypothesis: "t1 equals t2," which we do not preface with the "is provable that" notation. This expression is called a "side-condition." It is an additional constraint that must be satisfied in order for the inference rule schema to be instantiated.

We'll use this rule several times throughout the remainder of the lesson to illustrate some key concepts.

Soundness



A type system is sound iff

whenever 1. `A |- e : t` and

2. If `A(x) = t'`, then `x` has a value `v'` in `t'`

then `e` evaluates to a value `v` in `t`

If we can successfully type the if-then-else statement, then the first hypothesis in the rule guarantees that e0 will be a boolean; the latter hypotheses and side condition guarantee that e1 and e2 will be of the same type.

These sorts of guarantees are characteristic of sound type systems. Formally, we define a type system to be sound if and only if:

Whenever it is provable under an environment A that e has type t, and furthermore, whenever the environment A maps variable x to type t', then x has a value in that type t', then e always evaluates to some value v in t.

Comments on Soundness

- Soundness is extremely useful
  - Program type-checks => no errors at runtime
  - Verifies absence of a class of errors
- This is a very strong guarantee
  - Verified property holds in all executions
  - "Well-typed programs cannot go wrong"

Sound analyses are extremely useful. If a program type-checks according to a sound type system, then no errors of the kind checked by the type system can arise in any execution of the program. We say that a sound analysis verifies the absence of a class of errors.

This is a very strong guarantee: the property verified holds in all executions of the program. This guarantee is encapsulated by Robin Milner's classic slogan, "Well-typed programs cannot go wrong."

Comments on Soundness

- Soundness comes at a price: false positives
- Alternative: use unsound analysis
  - Reduces false positives
  - Introduces false negatives
- Type systems are sound
  - But most bug finding analyses are not sound

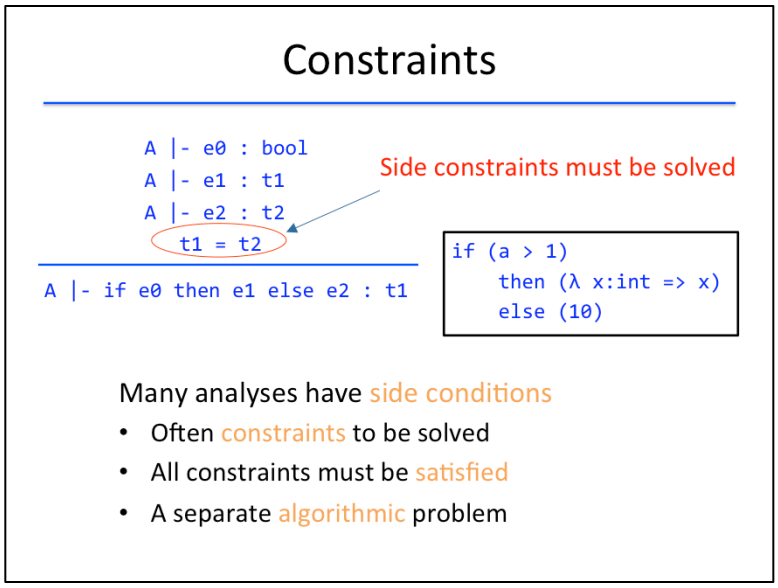
But soundness comes at a price. Because of the undecidability of static analysis, a sound analysis that always terminates must occasionally generate spurious errors or warnings -- that is, false positives -- due to the abstraction the analysis makes. This means that the sound analysis is incomplete.

An alternative approach is to use an unsound analysis, which allows us more control over the false positive rate. However, unsound analyses have the possibility of failing to detect errors -- that is, producing false negatives.

Type systems are sound analyses: while they may generate false positives on occasion, they will never produce a false negative. But most kinds of analyses for detecting bugs -- bug-finding analyses -- are not sound. You can learn more about these kinds of analyses for Java and C by following the links in the instructor notes.

[For Java: "Using static analysis to find bugs" http://ieeexplore.ieee.org/xpls/abs_all.jsp?arnumber=4602670&tag=1]

[For C: "A few billion lines of code later: using static analysis to find bugs in the real world" http://dl.acm.org/citation.cfm?id=1646374]



Let's look some more at the notion of side conditions. These conditions occur in many type-based analyses. They are constraints that must be solved -- that is, satisfied -- in order to verify that the inference rule is properly applied.

For example, consider the program "if (a > 1) then (lambda x:int => x) else 10" in an environment where a is of type int. Expression e1 in this program has type int -> int, while expression e2 has type int. Without the side-condition, this program would be typed as a function mapping ints to ints; this would cause the type system to be unsound, as the program would not evaluate to the correct type if `a` were ever less than or equal to 1.

Checking whether these side constraints can be solved is a separate algorithmic problem that we cover in the lesson on constraint-based analysis.

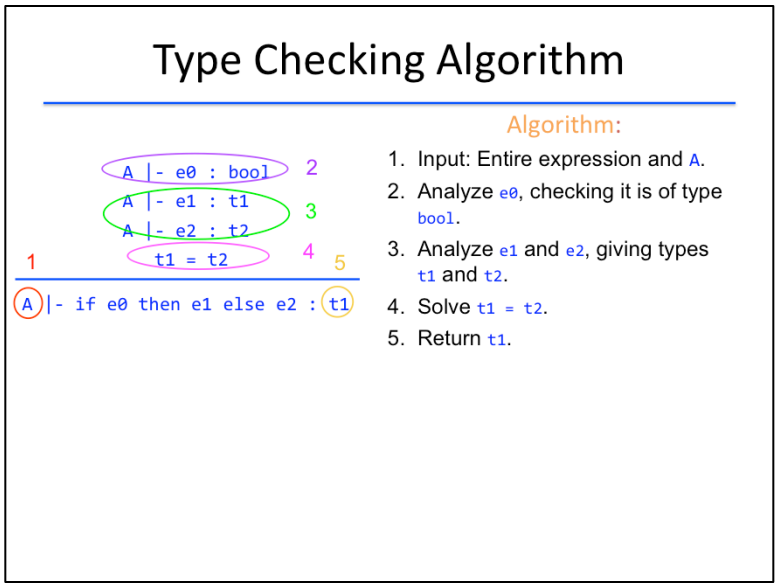
Another Example

- Consider a recursive function
  `f(x) = … f(e) …`
- If `x : t1` and `e : t2` then `t2 = t1`
  - Can be relaxed to `t2 ⊆ t1`
- Recursive functions yield recursive constraints
  - Same with loops
  - How hard constraints are to solve depends on constraint language, details of application

Another example of where side conditions occur are with recursive functions: functions which call themselves in their own definition. The constraint that must be satisfied is that if x is of type t1 and e is of type t2, then t2 equals t1. (In fact, this is a bit too strict; it could be relaxed to requiring only that t2 is a subset of t1.)

Recursive functions yield constraints that are themselves recursive. The same holds for loops. We observed this phenomenon in the lesson on dataflow analysis, wherein loops in programs necessitate multiple iterations of the chaotic iteration algorithm. That algorithm in reality is solving a system of recursive constraints, namely, dataflow constraints for computing the IN sets and OUT sets of dataflow facts at each program point.

In general, the difficulty of solving constraints depends on the language of the constraints and the details of the application.



We are now in a position to develop an algorithm to automatically type-check programs. The basic algorithm is as shown here.

The algorithm starts by taking as input the entire expression to be checked and the environment in which it will be checked.

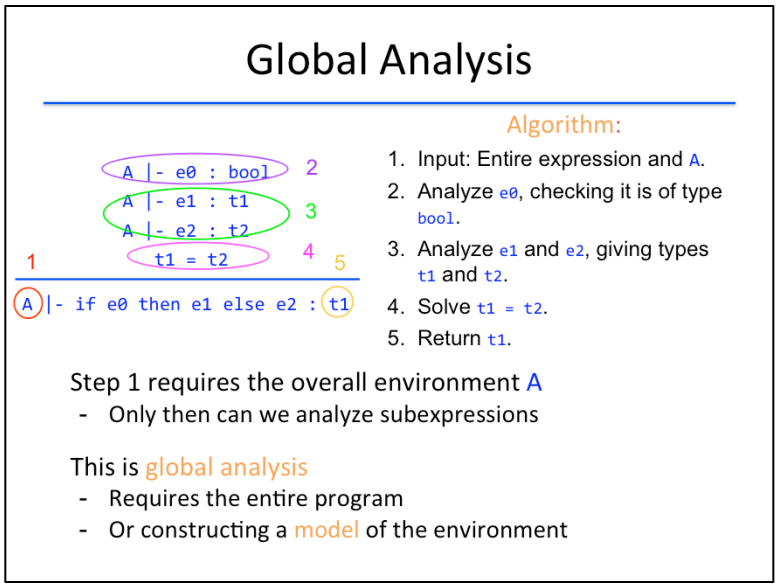
The algorithm then determines which schema to use in attempting to type the expression. Typically, there is a unique schema for each kind of expression, which is the case for our simple language. In this case, the syntactic structure of the expression uniquely determines which schema to use. For languages where this is not the case, the algorithm tries to type the expression using each applicable schema, and determines whether any schema succeeds.

If the schema is an inference rule schema, the algorithm recursively calls this procedure on each hypothesis of the schema.

The algorithm attempts to solve any side constraints.

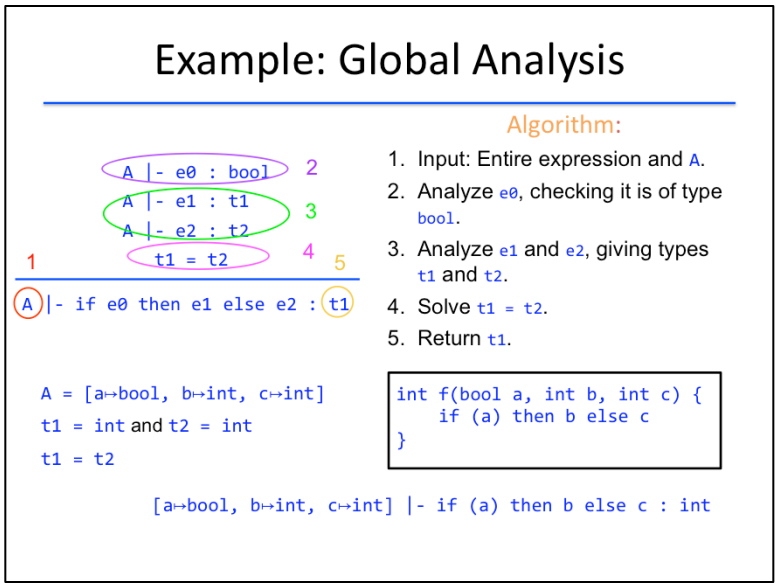
Assuming all hypotheses correctly type-check and side constraints are satisfied, the algorithm returns the appropriate type for the entire expression.

In this instance of the algorithm, we show how an if-then-else expression would be analyzed. First, e0 would be checked to ensure that it is of type bool. Then e1 and e2 would be analyzed to determine their types, t1 and t2 respectively, and the algorithm would attempt to satisfy the side-constraint t1 == t2. Assuming this constraint is satisfied, t1 would then be returned.



Notice that Step 1 requires the overall environment A, not just the particular expression being analyzed. Only then can we proceed to analyze the sub-expressions of the given expression. Intuitively, this environment A summarizes the result of the analysis of the remainder of the program in which this expression is a sub-expression.

This particular style of analysis is called a global analysis or a whole-program analysis, since it requires the entire program or providing a model of the environment. It is also called top-down analysis since it begins at top-level expressions and descends into sub-expressions.



Let's look at an example of global analysis in action. Consider the following program which consists of an if-then-else expression. We will show step-by-step how global analysis derives the following type judgement for typing this expression.

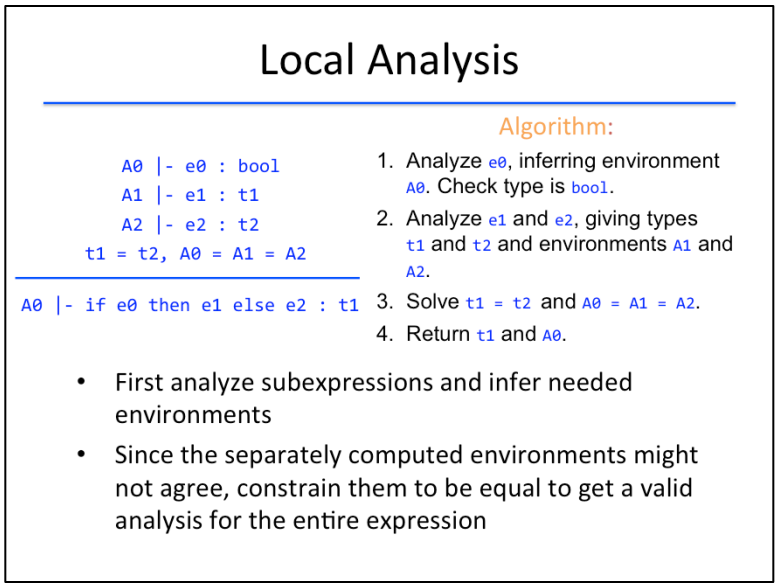
We start out with the input as this entire expression and this environment A which comes from the signature of function f. This environment summarizes the result of the analysis of the rest of the program in the following sense: whenever f is called from the rest of the program, a can be assumed to be of type boolean, and b and c to be of type integer.

In step 2, under this environment, we analyze e0, which in our example is the variable a, checking that it is of type boolean.

In step 3, under the same environment, we analyze e1 and e2, which in our example are variables b and c, giving types t1 and t2 as int and int, respectively.

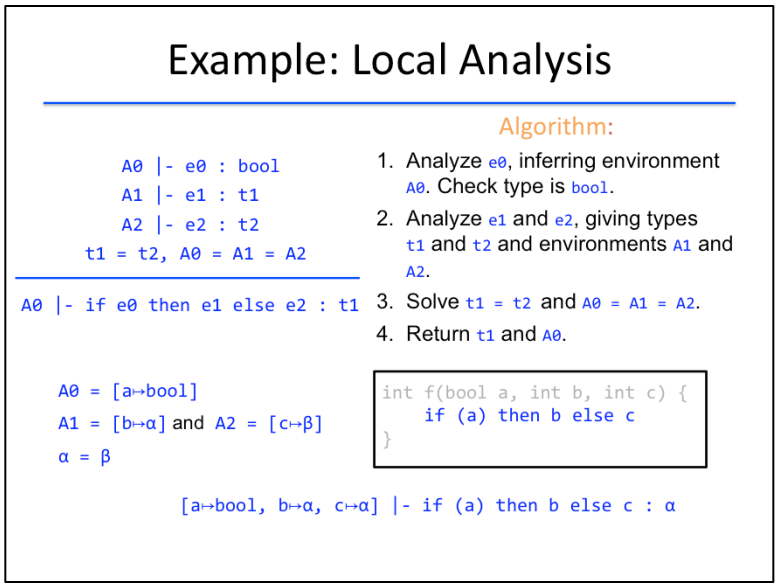
In step 4, the analysis solves side condition t1 = t2, which indeed holds, since it just inferred that both t1 and t2 are int.

Finally, the analysis returns t1, which is int.



In contrast to global analysis, there is local analysis, also called compositional analysis or bottom-up analysis. In this style of analysis, we first analyze sub-expressions, inferring the needed environments from the sub-expressions themselves. Since the separately computed environments might not agree, we constrain them to be equal. This allows us to generate a valid analysis for the entire expression.

For example, in analyzing the [If-Then-Else] schema, we would not pass in an environment to the algorithm. Instead, the algorithm would analyze e0, inferring an environment A0 from this analysis (as well as checking that e0 is a bool). Then e1 and e2 would be analyzed, generating two more inferred environments A1 and A2 and obtaining types t1 and t2 for the expressions. Finally, the side conditions t1 == t2 and A0 == A1 == A2 would need to be satisfied. If the algorithm is able to satisfy these constraints, then it returns the type t1 of the expression in the conclusion as well as the environment A0 needed to prove that the expression is of type t1.



Let's use the same example program to illustrate how local analysis operates. We have faded out the signature of function f to emphasize the fact that local analysis does not require a model of the environment. We will show step-by-step how local analysis derives the following type judgement for typing this expression.

In Step 1, we analyze e0, which in our example is the variable a. We infer this environment A0 necessary for checking that a is of type boolean.

In Step 2, we analyze e1 and e2, which in our example are variables b and c, giving types alpha and beta and environments A1 and A2 under which these variables have those corresponding types. Notice that, unlike in global analysis, where both these types were inferred to be int, this time we have unconstrained type parameters alpha and beta.

In Step 3, the analysis solves side condition t1 = t2, which constrains alpha and beta to be equal. The analysis also solves side condition A0 = A1 = A2, obtaining the following environment.

Finally, the analysis returns t1, which is alpha, and A0, which is this inferred environment. Notice that the resulting typing is more flexible than what global analysis inferred; in particular, the expression has an unconstrained type alpha, instead of int in the case of global analysis.

Global vs. Local Analysis

- Global Analysis:
  - –Usually technically simpler than local analysis
  - –May need extra work to model environments for unfinished programs
- Local Analysis:
  - –More flexible in application
  - –Technically harder: Need to allow unknown parameters, more side conditions

Why do we have two kinds of type analyses? As usual, there are tradeoffs between the two approaches that should be considered when deciding which one to apply.

Global analysis is usually simpler than local analysis as it assumes a model of the environment instead of inferring it itself. The drawback is it may require a lot of extra engineering to construct models of the environment for partial programs.

On the other hand, local analysis is more flexible: it can allow analysis of, for example, a library without a client. However, local analysis is harder to do in the following sense: it requires the algorithm to allow for unknown parameters in environments, which will be solved for later, such as alpha in the example we saw; and more side conditions on such unknown parameters that greatly increase the complexity of the problem.

QUIZ: Properties of Type Systems

Check the below untypable programs that can "go wrong":

`42 (λ x:int => (x + 5))`	☐
`(λ x:int => x) + 1`	☐
`if (true) then 1 else ((λ x:int => x) + 1)`	☐

`(if (c != 0) then (λ x:int => x)`
`else (λ x:int->int => (x 1)))`
`(if (c != 0) then 1`
`else (λ z:int => z))`
	☐

{QUIZ SLIDE}

We saw that type systems are sound in that well-typed programs do not go wrong in any execution. But type systems are incomplete, that is, untypeable programs do not necessarily go wrong in some execution.

Let's wrap up our discussion of the properties of type systems with a quiz that illustrates the incompleteness of the type system we designed for our simple language.

Consider the following four programs:

- 42 applied to the function (lambda x of type int yields x plus 5)
- the function (lambda x of type int yields x), plus one
- if true then 1 else (lambda x of type int yields x), plus one
- if (c != 0) then (lambda x of type int yields x), else (lambda x of type int -> int yields (x applied to 1), applied to
  if (c != 0) then 1, else (lambda z of type int yields z)

All of these programs are untypeable according to the rules we've described in this lesson. However, not all of them correspond to programs that would exhibit an error on execution. Check the boxes corresponding to which of these programs can "go wrong": that is, there is some execution on which an error would occur.

QUIZ: Properties of Type Systems

Check the below untypable programs that can "go wrong":

`42 (λ x:int => (x + 5))`	x
`(λ x:int => x) + 1`	x
`if (true) then 1 else ((λ x:int => x) + 1)`	

`(if (c != 0) then (λ x:int => x)`
`  else (λ x:int->int => (x 1)))`
`(if (c != 0) then 1`
`  else (λ z:int => z))`	

{SOLUTION SLIDE}

Let's consider each program in turn.

The first program is untypable because no rule allows for calling an integer like a function. Moreover, on every run of the program, this call to the integer would be encountered during execution, and it is undefined what should happen upon such a call. Therefore, this program "goes wrong" on all runs.

The second program is untypable again because none of our schemas allows for the addition of a function to an int. And, on every run of the program, this ill-defined addition would be encountered. So this program is another that could---and indeed always would---"go wrong."

The third program is also untypable because the else clause contains a function added to an int. However, notice that no execution of this program would ever encounter this ill-defined addition, as the boolean expression always evaluates to true. This shows that our type system is incomplete: even though the program would always evaluate to the int value 1 and never encounter the "bad" code, we still reject it as ill-typed.

QUIZ: Properties of Type Systems

Check the below untypable programs that can "go wrong":

`42 (λ x:int => (x + 5))`	☒
`(λ x:int => x) + 1`	☒
`if (true) then 1 else ((λ x:int => x) + 1)`	☐

`(if (c != 0) then (λ x:int => x)`
`else (λ x:int->int => (x 1)))`
`(if (c != 0) then 1`
`else (λ z:int => z))`
	☐

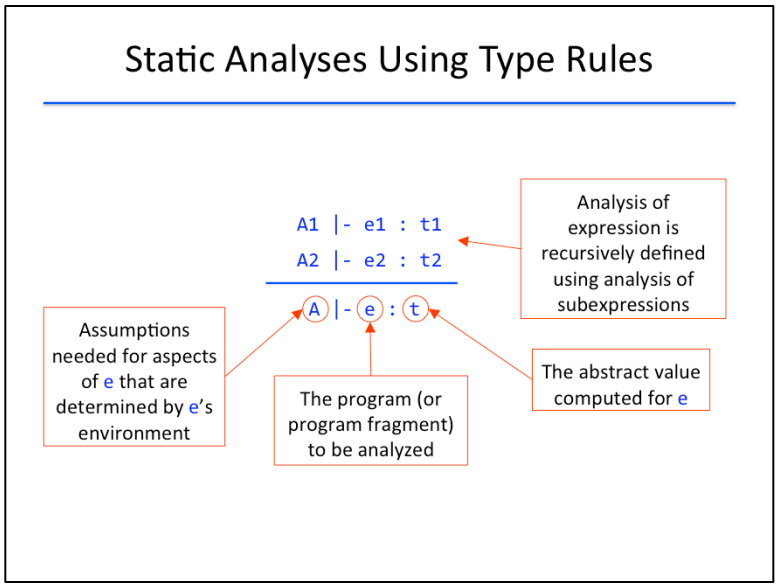
{SOLUTION SLIDE}

The fourth program is untypable in our system because we require "then" and "else" clauses of the same boolean expression to evaluate to the same type; however, the functions output by the first boolean expression have different domains: int -> int and (int -> int) -> int, respectively. Similarly, the outputs of the second boolean expression are an int and int -> int.

Nevertheless, even though this program isn't typable, we can see that on every run of the program, no ill-defined behavior would be encountered. If c is nonzero, then the first boolean expression would output a function mapping ints to ints, the second boolean expression would output 1, and we would then have the application of the function to 1.

On the other hand, if c were to equal 0, the first boolean expression would output a function which takes a function mapping ints to ints, and returning an int,
while the second boolean expression would output a function mapping ints to ints, and so the resulting function application would also be well-defined.

This gives us two examples -- two false positives -- where our type system is overly conservative about what it considers to be a valid program. While our type system can be proven to be sound, we will never be without these false positives, even if we try to introduce new rules allowing us to detect whether the "then" or "else" branches of boolean expressions are followed.



Earlier in the lesson, we promised that type notation would be useful in contexts aside from type systems. Indeed, they are useful for describing a wide range of static analyses. Instead of speaking in the language of types, we can interpret each part of a rule of inference in a different way:

We will use e to represent the program (or program fragment) to be analyzed.

t will represent an abstract value (which need not necessarily be a type) computed for the program e.

Instead of mapping variables to types, the environment A may now describe assumptions needed for aspects of e that are determined by the rest of the program.

This will allow us to recursively analyze hypotheses of each inference rule -- subexpressions of the program -- to prove that the program has some particular global property.

What sort of properties can we reason about in this manner?

An Example: The Rule of Signs

- Goal: to estimate the sign of a numeric computation
- Example: -3 * 4 = -12
- Abstraction: - * + = -

As a starting example, let's use type notation to determine the sign of a numeric computation. For example, we would like to prove that the output of -3 times 4 is negative. Abstracting this particular computation, we can represent it as "negative times positive equals negative."

Abstract Values

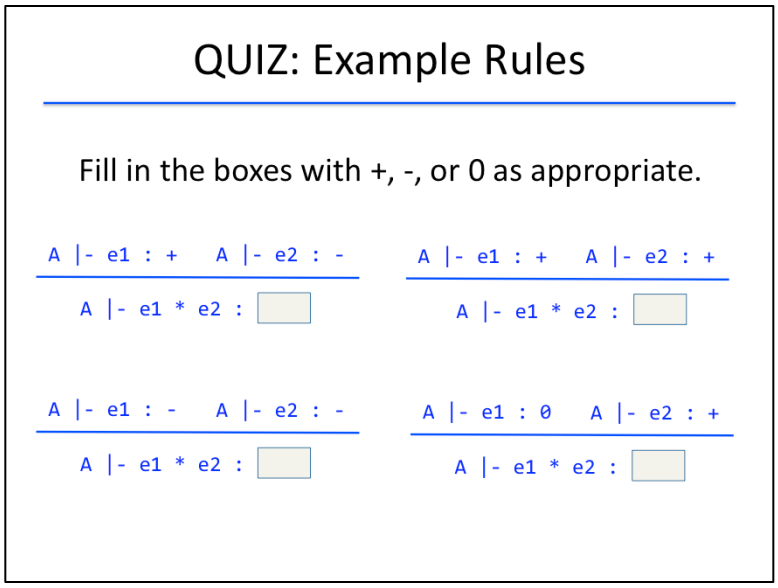
- + = { all positive integers }
- 0 = { 0 }
- - = { all negative integers }

- Environment A : Variables -> { +, 0, - }

The abstract values that we will use in this analysis are:

- PLUS, representing the set of all positive integers;
- ZERO, representing the set consisting of just the integer 0; and
- MINUS, representing the set of all negative integers.

The environment in this system will be a mapping from variables to one of these three abstract values.

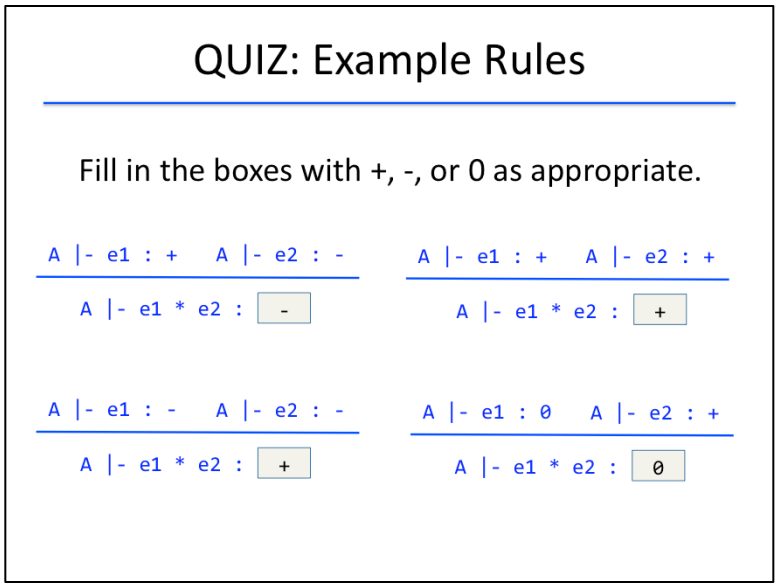


{QUIZ SLIDE}

Let's do a quiz on writing some example rules of inference that we would want to add to our analysis.

- Under the environment A, if it is provable that e1 has value PLUS and e2 has value MINUS, then it is provable under A that e1 times e2 has value BLANK.
- Under the environment A, if it is provable that e1 has value PLUS and e2 has value PLUS, then it is provable under A that e1 times e2 has value BLANK.
- Under the environment A, if it is provable that e1 has value MINUS and e2 has value MINUS, then it is provable under A that e1 times e2 has value BLANK.
- Under the environment A, if it is provable that e1 has value ZERO and e2 has value PLUS, then it is provable under A that e1 times e2 has value BLANK.

As a quick check of your understanding, enter the appropriate abstract value for the conclusion of each inference rule. Enter either the plus symbol, the minus symbol, or the number 0.



{SOLUTION SLIDE}

Recall that a positive number times a negative number should be a negative number, so:

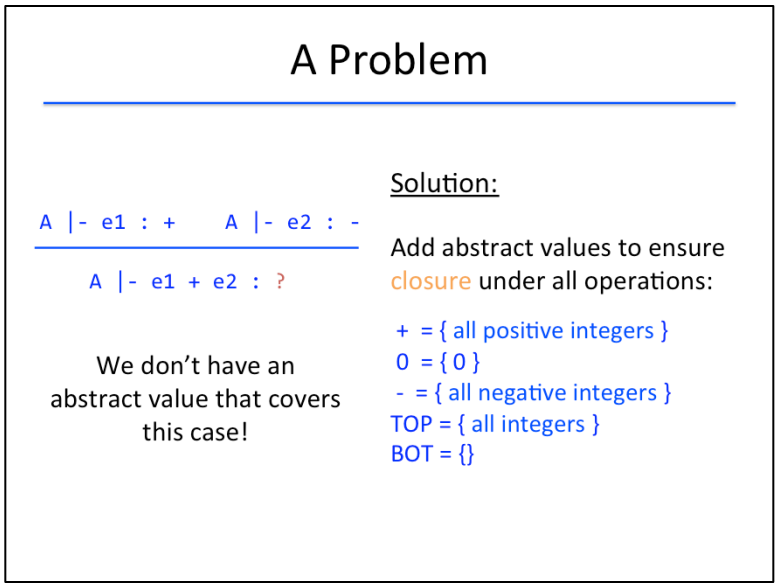
Under the environment A, if it is provable that e1 has value PLUS and e2 has value MINUS, then it is provable under A that e1 times e2 has value MINUS.

Further, a positive number times a positive number should be positive, so:
Under the environment A, if it is provable that e1 has value PLUS and e2 has value PLUS, then it is provable under A that e1 times e2 has value PLUS.

Similarly, a negative times a negative is positive, so:
Under the environment A, if it is provable that e1 has value MINUS and e2 has value MINUS, then it is provable under A that e1 times e2 has value PLUS.

Finally, zero times any number is zero, so:
Under the environment A, if it is provable that e1 has value ZERO and e2 has value PLUS, then it is provable under A that e1 times e2 has value ZERO.

Note that these don't constitute all possible rules we would want to add: we would need to add more to fully type all possible expressions (not to mention that we would need to add axioms like [Int] and [Var] from our lambda-calculus language).



So far, we have described rules for deriving the sign of the product of two expressions. However, what if we want to describe the sign of the sum of two expressions? Let's add the following rule:

Under environment A, if it is provable that e1 has value PLUS and e2 has value MINUS, then it is provable under A that e1 plus e2 has value ... what?

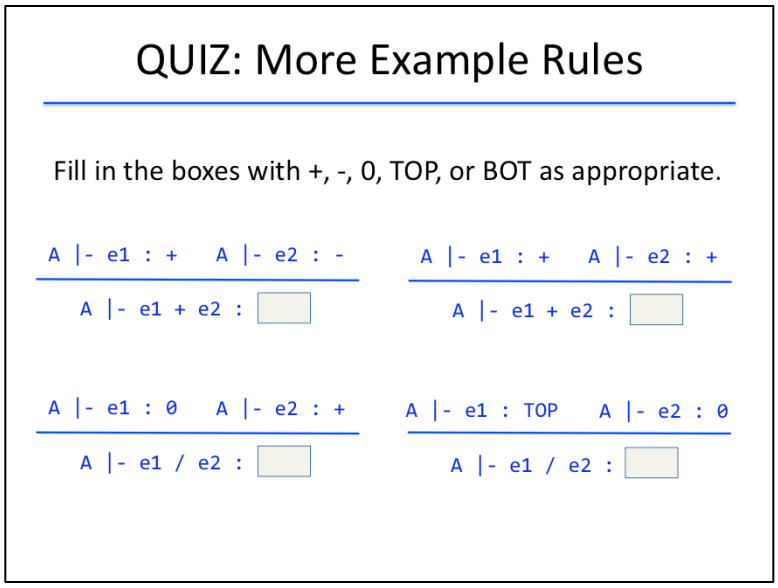
We have a problem here. We don't have an abstract value that covers this case.

How do we solve this problem? The solution is to add new abstract values to our system to make sure the system is closed under all operations: there are no expressions with undefined abstract values.

To our existing abstract values, we will add two additional abstract values:

- TOP, which will represent the set of all integers
- BOT, which will represent the empty set

Notice that the abstract values no longer represent disjoint sets of concrete values: this is OK! This happens quite regularly in real-world programming. A subclass in an object-oriented language is an abstract value that represents a subset of concrete values represented by its superclass. Similarly, all of the abstract values in this example represent subsets of the abstract value TOP and supersets of the abstract value BOT.



{QUIZ SLIDE}

For this quiz, let's try to determine the appropriate abstract value for the conclusion in each of the following inference rules:

- Under the environment A, if it is provable that e1 has value PLUS and e2 has value MINUS, then it is provable under A that e1 plus e2 has value BLANK.
- Under the environment A, if it is provable that e1 has value PLUS and e2 has value PLUS, then it is provable under A that e1 plus e2 has value BLANK.
- Under the environment A, if it is provable that e1 has value ZERO and e2 has value PLUS, then it is provable under A that e1 divided by e2 has value BLANK.
- Under the environment A, if it is provable that e1 has value TOP and e2 has value ZERO, then it is provable under A that e1 divided by e2 has value BLANK.

Notice that we're also introducing rules of inference about division now: just use standard rules of arithmetic to determine your answers here.

Fill in each of the boxes with a plus, minus, zero, the word TOP, or the word BOT as appropriate.

QUIZ: More Example Rules

Fill in the boxes with +, -, 0, TOP, or BOT as appropriate.

$$\frac{A \vdash e1 : +\quad A \vdash e2 : -}{A \vdash e1 + e2 : \boxed{\text{TOP}}}$$

$$\frac{A \vdash e1 : +\quad A \vdash e2 : +}{A \vdash e1 + e2 : \boxed{+}}$$

$$\frac{A \vdash e1 : 0\quad A \vdash e2 : +}{A \vdash e1 / e2 : \boxed{0}}$$

$$\frac{A \vdash e1 : \text{TOP}\quad A \vdash e2 : 0}{A \vdash e1 / e2 : \boxed{\text{BOT}}}$$

{SOLUTION SLIDE}

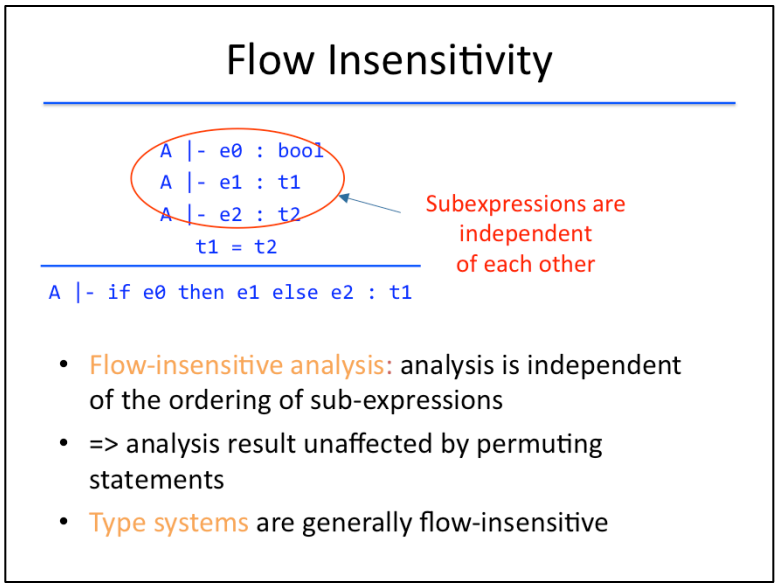
Let's start with the problematic example we considered earlier: if we add a positive number to a negative number, we could end up with a positive, negative, or zero sum. Therefore,
Under the environment A, if it is provable that e1 has value PLUS and e2 has value MINUS, then it is provable under A that e1 plus e2 has value TOP.

However, if we add two positive numbers, the outcome will definitely be a positive number. Therefore,
Under the environment A, if it is provable that e1 has value PLUS and e2 has value PLUS, then it is provable under A that e1 plus e2 has value PLUS.

Next, we consider division of zero by a positive number. This will definitely always output zero, so
Under the environment A, if it is provable that e1 has value ZERO and e2 has value PLUS, then it is provable under A that e1 divided by e2 has value ZERO.

On the other hand, division of any number by zero is undefined, so
Under the environment A, if it is provable that e1 has value TOP and e2 has value ZERO, then it is provable under A that e1 divided by e2 has value BOT.

These rules would, of course, need to be augmented with several more to express the entire sign computation analysis. As an exercise, try to write down the rules to extend this system to the four basic arithmetic operations: addition, subtraction, multiplication, and division. You can even try to add the modulus operation if you like!



Backtracking a bit, let's return to our analysis of the if-then-else expression from our lambda-calculus language. (It may seem like a digression from our current topic, but it will wrap back up nicely.)

Recall that in our algorithm's examination of the if-then-else expression, each of the subexpressions were analyzed independently of one another. It didn't take into account the order in which the expressions were listed; it only checked that e0, e1, and e2 were of the appropriate types. We could have swapped e1 and e2, and the analysis would be unaffected.

This is called flow-insensitive analysis, which you may recall from our study of pointer analysis. The analysis is independent of the order in which the statements appear in the program; in effect, it treats the program as a set of statements with little other structure imposed.

In practice, you'll find that most type systems are flow-insensitive.

Comments on Flow Insensitivity

- Flow insensitive analyses are often very efficient and scalable
- No need for modeling a separate state for each subexpression
- But can be imprecise ...

The reason for disregarding the control flow of the program is the usual reason for abstraction in static analysis: it simplifies the representation of the program state that our algorithm operates on. In particular, there is no need for modeling a separate state for each subexpression.

The resulting analyses are efficient and scalable, allowing them to be used liberally in many applications.

The tradeoff is loss of precision -- the usual tradeoff for speed in static analysis. To attain a faster, sound type system, we necessarily end up generating more false positives. (Indeed, we have a constant time type-checking algorithm that's perfectly sound: just reject all programs! But of course such a scheme, while fast and sound, is completely useless.)

Flow Sensitivity

```
A |- e0 : bool ▷ A0
A0 |- e1 : t1   ▷ A1
A0 |- e2 : t2   ▷ A2
  t1 = t2, A1 = A2
-------------------------------
A |- if e0 then e1 else e2 : t1 ▷ A1
```

Rules produce new environments, and analysis of a subexpression cannot happen until its environment is available

- Flow-sensitive analysis: analysis of subexpressions ordered by environments
- => analysis result depends on order of statements
- Dataflow analysis is an example of a flow-sensitive analysis

If we are willing to sacrifice simplicity and efficiency in order to increase our precision, we can make our abstraction of the program state more realistic. One way to do so is to remain sensitive to control flow.

In this case, each statement has an “input” environment and an “output” environment, and we order the analysis of subexpressions by the environments that are produced. Therefore, the result of the analysis now depends on the order of the statements, allowing us to make a finer distinction between correctly-typed and incorrectly-typed statements.

In the [If-Then-Else] schema example, we would start by proving that under environment A, e0 is a bool, and then output a new environment A0. This environment then flows into the analysis of both the statement that e1 is of type t1 and that e2 is of type t2, each of which produce their own “output” environments A1 and A2. We would then need to solve the side condition that environments A1 and A2 are equal (in addition to the previous side condition that t1 = t2) in order to prove that, under the environment A, the expression “if e0 then e1 else e2” has type t1 and output environment A1.

This kind of analysis is called flow-sensitive analysis.

Now is where we start tying together our previous thread---on different types of static analysis---with this thread. The “input” and “output” environments in this kind of static analysis should remind you of something we saw earlier: in dataflow analysis, we had “in” and “out” sets of program facts. These sets would play the part of the input and output environments. Therefore, dataflow analysis (as we might expect from its name) is a flow-sensitive static analysis which could be represented using type-system-like notation.

Comments on Flow Sensitivity

- Example:
  - Rule of signs extended with assignment statements

$$\frac{A \vdash e : + \triangleright A}{A \vdash x := e \triangleright A[x \mapsto +]}$$

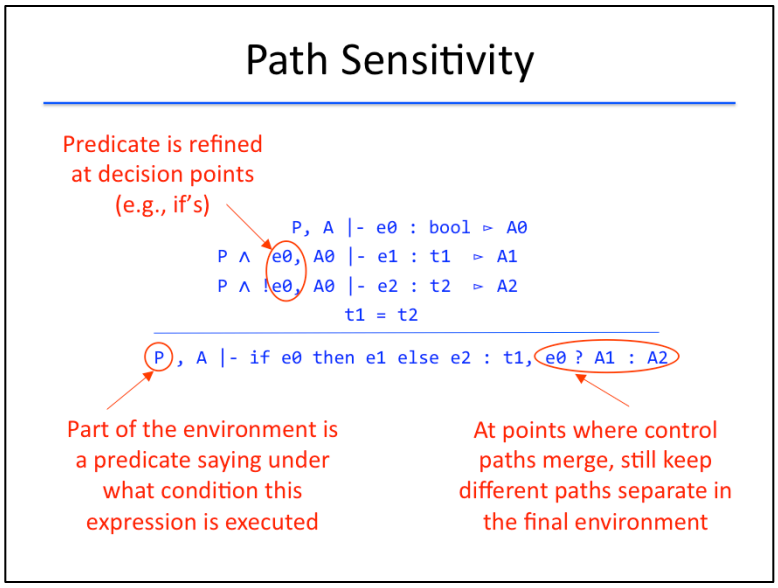
  - $A[x \mapsto +]$ means A modified so that $A(x) = +$
- Flow-sensitive analysis can be expensive
  - Each statement has own model of state
  - Polynomial cost increase over flow-insensitive

The environments shown previously in flow-sensitive analysis were fairly abstract. Let's look at a more concrete example by extending our rule-of-signs system with assignment statements.

We add the following inference rule schema:

Under environment A, if it is provable that e has value PLUS and results in environment A, then, under environment A, it is provable that x assigned the expression e results in the environment A, modified so that A maps x to the value PLUS.

As we alluded to earlier, flow-sensitive analysis can be expensive. Our abstraction is more complex: each statement has to maintain its own separate model of the program's state. Because we are keeping track of more information, we end up with a polynomial increase in cost over flow-insensitive analyses.



We can further refine our model of the program's state in order to improve the precision of our analysis. Recall that, for the statement "if e0 then e1 else e2," our analysis doesn't consider whether e0 is true or false. This leads to situations where dead code (that is, if e0 is always true or always false) can cause the program to fail to type-check, or to a situation where the type system rejects a program that would have ill-defined behavior on a path that is logically impossible to take through the code. (We saw both of these situations in a previous quiz.)

We call this sort of analysis "path-insensitive," as it does not consider whether certain paths through the code are impossible to take; it rejects a program that has a "bad" path, even if this path can never be taken (yielding false positives in the process). By making our analysis sensitive to program paths, we are able to eliminate this form of false positive at the cost of more complexity in our program abstraction.

To do so, we augment our environment for each statement with a boolean predicate P. If P is the predicate under which an [If-Then-Else] statement would be executed, then we add P to the input environment of the first hypothesis (that e0 is a bool). For the second hypothesis (that e1 has type t1), we add P conjoined with e0 to the input environment. And for the third hypothesis (that e2 has type t2), we add P conjoined with not-e0 to the input environment. (Note that we are still retaining the distinction between input and output environments for all the hypotheses.)

Additionally, we no longer solve the side constraint that the output environments A1 and A2 are equal; instead, the conclusion allows for either of these to be its output environment based on the truth value of e0.

Comments on Path Sensitivity

- Symbolic execution is an example
  - Path-sensitive analyses also flow-sensitive
- Can be expensive but a necessary evil
  - Exponential number of paths to track
- Often implemented with backtracking
  - Explore one path
  - Backtrack to a decision point, explore another path

Symbolic execution, which we will study later, is an example of a path-sensitive analysis. (Note that path-sensitive analyses are also flow-sensitive, as we preserve the input-output environment ordering.)

As you might imagine, keeping track of these different execution paths can be quite expensive. In the general case, there can be an exponential number of paths to track compared to the number of statements in the program! But this complexity appears to be a necessary evil in many different applications.

Path-sensitive analyses are often performed using backtracking instead of explicitly merging environments together. A single path is explored until termination, and then the analysis backtracks to a decision point before exploring another path. This allows clients to prioritize the exploration of paths that are potentially more interesting than others, for instance, in the case of a bug-finding tool, paths that are more likely to expose a bug.

QUIZ: Flow and Path Sensitivity

For each program, select the kinds of analyses that can verify the indicated property:

Program	Property	Flow-insensitive	Flow-sensitive	Path-sensitive
`x = "a"; y = 5; z = 3+y; w = x+"b"`	No int plus string errors	☐	☐	☐
`x = 5; y = 1 / x; x = 0`	No divide-by-zero errors	☐	☐	☐
`if (y != 0) then 1 / y else y`		☐	☐	☐
`acquireLock(r); releaseLock(r)`	Correct locking	☐	☐	☐
`if (z > 0) then acquireLock(r);` `if (z > 0) then releaseLock(r)`		☐	☐	☐

{QUIZ SLIDE}

In this quiz, you will decide whether some sample programs can be proven to have certain properties by flow-insensitive, flow-sensitive, and path-sensitive analyses. For each of the following five programs, check the boxes corresponding to the kinds of analyses that can verify the specified property.

The first program consists of four statements executed in the following order: x is assigned the string literal "a"; y is assigned 5, z is assigned the result of 3 + y, and w is assigned the result of x + the string literal "b". For this program, the property to verify is that there are no integer plus string errors. For this question, allow the analysis to assume that each variable is defined before being used, which is indeed the case for this program.

For the second and third programs, the property to verify is that there are no divide-by-zero errors. The second program consists of three statements executed in the following order: x is assigned 5; y is assigned 1 divided by x; and x is assigned 0. The third program consists of an if-then-else expression: if y does not equal zero, then 1 divided by y, else y. Note that the variable y may have any integer value at the start of the program.

For the next two programs, the property to verify is that locks are used correctly: that is, there is no attempt to acquire a lock on a resource already locked, and there is no attempt to release a lock on a resource that is unlocked. The fourth program consists of two statements executed in the following order: acquireLock(r); releaseLock(r). Allow the analysis to assume that r is not locked at the start of the program. Finally, the fifth program consists of two if-then statements executed in the following order: if (z > 0) then acquireLock(r); followed by if (z > 0) then releaseLock(r). Note that the second if-then statement is not nested within the first if-then statement. Also note that variable z can have any integer value at the start of the program. Finally, as for the preceding program, allow the analysis to assume that r is not locked at the start of the program.

QUIZ: Flow and Path Sensitivity

For each program, select the kinds of analyses that can verify the indicated property:

Program	Property	Flow-insensitive	Flow-sensitive	Path-sensitive
`x = “a”; y = 5; z = 3+y; w = x+“b”`	No int plus string errors	X	X	X
`x = 5; y = 1 / x; x = 0`	No divide-by-zero errors		X	X
`if (y != 0) then 1 / y else y`				X
`acquireLock(r); releaseLock(r)`	Correct locking		X	X
`if (z > 0) then acquireLock(r);` `if (z > 0) then releaseLock(r)`				X

{SOLUTION SLIDE}

The first program can be proven to be free of integer plus string errors using a flow-insensitive analysis, assuming that all variables are defined in the program before being used.
Let’s see why this is the case. A flow insensitive analysis views this program as an unordered set of these four statements, and effectively checks all possible orderings of these four statements for integer-plus-string errors. The analysis ignores orderings where a variable is used before being defined, since we allowed the analysis to make this assumption. Then, it is easy to see that on each remaining ordering, the analysis verifies that there is no integer-plus-string error. Since the original program represents one of these orderings, if follows that the flow-insensitive analysis has proven that it too is free of integer-plus-string errors.

Since flow-sensitive and path-sensitive analyses are strictly more powerful than flow-insensitive analysis, it follows that they too can prove this program free of integer-plus-string errors.

The second program cannot be proven by flow-insensitive analysis to be free of divide-by-zero errors. Because flow-insensitive analysis considers the statements as an unordered set, it does not know that the assignment of 0 to x comes after the assignment of 1/x to y. However, flow-sensitive analysis does consider statement order, so it will know that x will not be 0 at the execution of the division operation. Because path-sensitive analysis is strictly more powerful than flow-sensitive analysis, it will also be able to verify that the program will not throw a division-by-zero error.

QUIZ: Flow and Path Sensitivity

For each program, select the kinds of analyses that can verify the indicated property:

Program	Property	Flow-insensitive	Flow-sensitive	Path-sensitive
`x = "a"; y = 5; z = 3+y; w = x+"b"`	No int plus string errors	X	X	X
`x = 5; y = 1 / x; x = 0`	No divide-by-zero errors		X	X
`if (y != 0) then 1 / y else y`				X
`acquireLock(r); releaseLock(r)`	Correct locking		X	X
`if (z > 0) then acquireLock(r);` `if (z > 0) then releaseLock(r)`				X

{SOLUTION SLIDE}

The third program cannot be proven even by flow-sensitive analysis to be free of divide-by-zero errors. Flow-sensitive analysis, while it does keep track of the order of statement execution, does not keep track of which branches are taken at any given decision point for the program. For example, if 1/y and y had been switched so that the program read "if y does not equal 0, then y, else 1/y", a path-insensitive analysis could not tell the difference. All it would know is that either y or 1/y could appear after the check that y doesn't equal zero. Therefore, it could not prove that the division by y won't throw an error. However, a path-sensitive analysis would be able to tell the difference if we were to swap the positions of 1/y and y in the if-then-else expression. In particular, it could prove that if 1/y is in the y != 0 branch, then an error would not occur.

Now, for the fourth and fifth programs, the analysis must prove that the program does not acquire any locked resources or release any unlocked resources.

The fourth program relies on the particular statement execution order in order to ensure that the lock on r is not released while r is unlocked. Therefore, a flow-insensitive analysis could not prove that r is not improperly unlocked, while a flow-sensitive analysis (and therefore a path-sensitive analysis) could.

To verify that the fifth program does not have any locking errors, certainly requires at least flow sensitivity to detect that any lock acquisition comes before lock release. However, just flow-sensitivity is not sufficient in this case, as it is too weak to recognize that if z > 0, the "then" portion of both branch points must be taken. Path-sensitive analysis would be able to recognize this fact, so path-sensitive analysis is able to prove that the program does not have any locking errors.

Summary

- Very rough taxonomy:
  - Type systems = flow-insensitive
  - Dataflow analysis = flow-sensitive
  - Symbolic execution = path-sensitive
- Lines have been blurred
  - Many flow-sensitive type systems and path-sensitive dataflow analyses in research literature

We have discussed different kinds of analyses and their abilities to detect errors in programs with different structures. While we have used the notation and terminology of type systems in this discussion, the typical applications of these analyses often have different names and purposes in practical applications.

While it is certainly possible to create a type system that is flow-sensitive or even path-sensitive, we normally think of type systems as flow-insensitive analyses. Flow-sensitive analyses are usually associated with dataflow analyses, and symbolic execution is a prototypical path-sensitive analysis.

However, these lines have become more blurred lately. For example, in the research literature, it is not uncommon to come across flow-sensitive type systems and path-sensitive dataflow analyses.

What Have We Learned?

- What is a type
- Computing types of programs using type rules
- Properties of type systems: soundness, incompleteness, global vs. local type checking
- Describing other analyses using types notation
- Classifying analyses: flow-insensitive vs. flow-sensitive vs. path-sensitive

This concludes our introduction to type systems and their usefulness to static program analysis in general.

In this lesson, we have learned:

- What a "type" is: an abstract value consisting of a set of concrete values having that type
- How to compute the types of programs using derivations from type rules
- The properties of type systems: soundness (that is, guaranteeing no false negatives), incompleteness (that is, the occurrence of false positives), and global vs. local type-checking
- How to describe other kinds of static analyses using the notation of type systems, and
- The strengths and costs of flow-insensitive, flow-sensitive, and path-sensitive analyses.

We have only scratched the surface of type systems in this lesson. The study of type systems is an active area of research with many fruitful results. We have included a list of resources in the instructor notes for further reading on topics in type systems.

[Article titled "Type Systems" by Luca Cardelli in CRC Handbook:
http://www.cc.gatech.edu/~naik/courses/cs6340/papers/cardelli-types.pdf]

[Book titled "Types and Programming Languages" by Benjamin Pierce:
https://www.cis.upenn.edu/~bcpierce/tapl/]

[Book titled "Advanced Topics in Types and Programming Languages"
edited by Benjamin Pierce: https://www.cis.upenn.edu/~bcpierce/attapl/]